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Nash's Bargaining Formula Revisited:
A Note on Self-Referential Logic

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Abstract

The note focuses on the marginal rates of substitution (MRS) in Nash's product formula solution to bargaining and why the formula works. Two simple examples from duopoly and bilateral monopoly are used to demonstrate that the MRS's for both players are implicitly in the contract curve and the product formula. They are equal in the former by design. They become equal in the latter in equilibrium. The self-referential logic is evident. The bargaining model or system is self-contained and circular and is analogous to the proposition given by x = F(x).

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In a now famous paper, John Nash (1950) demonstrated that given the assumptions of invariance of utility functions, Pareto Optimum, symmetry, and independence of irrelevant alternatives, the equilibrium solution to a bargaining problem will be given by the maximization of the product of the utilities of the two players in a cooperative game. And, further, the product formula is the only formula consistent with these assumptions (see, Luce and Raiffa, 1957, for this rendition). In mathematical logic I attempt to show that the demonstration is analogous to a self-referential statement of the type that $x = F(x)$ as used, for example, by Kurt Gödel (as discussed in Goldstein, 2005, pp. 177-85). If the four assumptions by virtue of the product formula map into themselves the system is closed and circular. The assumptions imply the product formula and the product formula implies the assumptions.

In this brief note, the focus is on the Pareto Optimum condition as defined by the equality of the marginal rates of substitution (MRS) of the players (see, Nash, 1953 and Mayberry, Nash, and Shubik, 1953). I demonstrate using as an example the traditional and simple Cournot-Nash duopoly bargaining problem in the output space that the product formula, namely, the product of the profit functions (utility functions), under maximization does indeed implicitly contain the Pareto Optimum assumption as defined by the equality of the MRS's of the two firms. Indirectly, the example also shows why/how the product formula works. This note is not a proof of the Nash assertion. Nash and others have done this amply. In the context of the simple duopoly game, the equality of the MRS's is given by the vanishing determinant of the Jacobian matrix for the two profit functions.

It is convenient to show the MRS form of the Pareto Optimum condition first. In general terms, the determinant of the Jacobian for the two profit functions is given by

(1) $$|J| = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (ad - bc) = 0,$$

where the profit for each firm depends on its output and its rival's output (q_1 and q_2). The right-hand form of the determinant gives the Edgeworth contract curve (CC). Upon rearranging the right-hand form, the equality of the MRS's is obtained and given by

(2) $a/c = b/d = dq_2/dq_1 = MRS_i$ $(i = 1, 2)$.

To show (2) explicitly for the simple example, let the market demand be $P = 1 - q_1 - q_2$, costs be zero, $\Pi_1 = (1 - q_1 - q_2)q_1$, and , $\Pi_2 = (1 - q_1 - q_2)q_2$. Then,[1]

(3)) $$|J| = (1 - 2q_1 - q_2)(1 - q_1 - 2q_2) - q_1q_2 = 0$$

$$= (q_1M_1)(q_2\tilde{M}_2) - q_1q_2 = 0$$

$$= M_1\tilde{M}_2 - 1 = 0,$$

where $\tilde{M}_2$ is the inverse of MRS_2. Thus, $M_1/M_2 = 1$, and the MRS's are equal all along the CC (a well understood result). My point here is simply to show explicitly that the Jacobian determinant contains the Pareto Optimum condition, independently of the product formula.

Next, I demonstrate for the simple example that the product formula implicitly contains the two MRS's. Let the product of the two profit functions be given by

(4) $$W(q_1, q_2) = \Pi_1 \Pi_2 = [(1 - q_1 - q_2)q_1][(1 - q_1 - q_2)q_2]$$

$$= (1 - q_1 - q_2)(1 - q_1 - q_2)q_1q_2.$$

Insert dummy terms, $(2q_1 - 2q_1)$ and $(2q_2 - 2q_2)$ into the first and second products respectively to obtain

(5) $W = [(1 - 2q_1 - q_2 + q_1)(1 - 2q_2 - q_1 + q_2)]q_1q_2$

$$= [q_1(M_1 + 1)q_2(\tilde{M}_2 + 1)]q_1q_2$$

$$= [(M_1 + 1)(\tilde{M}_2 + 1)](q_1q_2)^2 ,$$

where as before $\tilde{M}_2$ is the inverse of the MRS_2 . Thus, the product formula contains the MRS's implicitly. However, they are not necessarily equal. They are only equal at the equilibrium point on the CC where the iso-W curve in the q-space is tangent to the CC.

To complete the demonstration, as is also well-known, the cartel solution for the simple example under joint-profit maximization is $Q = q_1 + q_2 = 1/2$ and $P = 1/2$. Joint profit is given by $PQ = 1/4 = \Pi_1 + \Pi_2$. The bargaining problem is to determine a "fair" (in Nash's sense) distribution of the maximum profit. Formally, the problem can be set up in Lagrangean form and given by[2]

(6) $L(q_1 , q_2 , \lambda) = \Pi_1 (.) \Pi_2 (.) + \lambda(1/2 - q_1 - q_2)$.

The first-order conditions after eliminating λ are set equal and given by

(7) $(1 - 2q_1 - q_2) (q_2 - q_2^2 - q_1q_2) + (q_1 - q_1^2 - q_1q_2)(- q_2) =$

$$(1 - 2q_2 - q_1) (q_1 - q_1^2 - q_1q_2) + (q_2 - q_2^2 - q_1q_2)(- q_1).$$

Again, by inserting the previously used dummy terms into the appropriate profit functions and noting the definitions of the MRS's , (7) after somewhat tedious rearrangements can be shown to be

(8) $q_2(M_1 + 1)(\tilde{M}_2 + 1) - q_1(M_1 + 1)(\tilde{M}_2 + 1) = 0,$

$$q_2 - q_1 = 0.$$

With $q_2 = q_1$ then from the constraint, both equal 1/4. The optimum distribution of the joint profit is thus $Pq_i = 1/8$ to each firm. It can also be shown that the equal MRS' are also equal to one in equilibrium for the simple data used.

The interesting point about this whole optimization process is that while the MRS's are equal and embedded in the CC and are also embedded in the product formula and thus in equilibrium are also equal, they are eliminated in the first-order conditions. This elimination is not a problem. The first-order conditions are simply one rule to determine the optimum outputs. The optimum outputs could just as easily be determined by a numerical iteration algorithm (like the simplex method in linear programming).

To summarize, the Jocobian has equal MRS's and the product formula implicitly contains the MRS's. At the tangency (equilibrium in the output space), the MRS's in the product formula are equal since they have a common point on the CC. In the simple example the MRS's are also equal to one. The Nash assertion is that the product formula satisfies the Pareto Optimum condition. How is this so? It implicitly contains the MRS's and satisfies the condition in equilibrium. The self-referential principle in logic referred to at the outset, is a function such that x = F(x), like the fixed-point theorem. Here, in the context of the demonstration given, x is the MRS's in W(.) and F is the mapping to the CC and thus to the MRS's given by x. So, the system is, in effect, self-contained and circular. The Jacobian equation from which the CC comes can stand alone. All along the CC the MRS's are equal. The product formula has the MRS's in it. So, it is a problem of matching the two functions both with the same common factor, namely, the MRS's. Any other method or formula for dividing the joint profit would not have this common factor.[3]

References

Friedman, J. W. *Game Theory with Applications to Economics*. New York, New York: Oxford University Press, 1986, pp. 179-80.

Goldstein, R. *Incompleteness: The Proof and Paradox of Kurt Gödel.* New York, New York: W. W. Norton & Co., 2005, pp. 175-85.

Luce, R. D. and Raiffa, H. *Games and Decisions: Introduction and Critical Survey*. New York, New York: John Wiley & Sons, Inc., 1957, pp. 124-37.

Mayberry, J. P., Nash, J. F., and Shubik, M. "A Comparison of Treatments of a Duopoly Situation." *Econometrica*, 1953, 21(1), pp. 141-54.

Nash, J. F. "The Bargaining Problem." *Econometrica*, 1950, 18(2), pp. 155-62.

Nash, J. F. "Two-Person Cooperative Games." *Econometrica*, 1953, 21(1), pp. 128-41.

Footnotes

[1]The total differential for firm 1's profit function $\Pi_1 = (1 - q_1 - q_2)q_1$ is $d\Pi_1 = dq_1 - 2q_1dq_1 - q_2dq_1 - q_1dq_2 = 0$. Upon rearrangement, $dq_2/dq_1 = (1 - 2q_1 - q_2)/q_1 = M_1$, the MRS for firm 1. In a similar way, $dq_2/dq_1 = q_2/(1 - 2q_2 - q_1) = M_2$, the MRS for firm 2. Note, however, that in the derivations in the text, the inverse of M_2 (namely, $\tilde{M}_2$) has to be used first to be consistent with the use of the dummy terms.

[2]The Jacobian determinant given by (2) for the simple example is $(1 - 3q_1 - 3q_2 + 4q_1q_2 + 2(q_1)^2 + 2(q_2)^2) = 0$. In spite of the non-linear appearance of this CC, its total differential shows it has a constant slope of -1. Thus, the constraint on (6) is given simply as linearly.

[3]A similar demonstration is possible using the firm-union bilateral monopoly example in Friedman (1986, pp. 179-80). With firm profit = $L(100 - L) - wL$ and union utility = $\sqrt{L}w$, where L is employment and w is the wage rate, the vanishing Jacobian determinant in the L – w space after rearrangements like in the text is $(MRS_f - MRS_u) = 0$. The $MRS_f = (100 - 2L - w)/L = dw/dL$ and the $MRS_u = -w/L = dw/dL$. The product of the two functions in the L – w space is $L^3(MRS_f + 1)(i)\sqrt{}\, MRS_u)$, where $(i) = \sqrt{-1}$. In equilibrium, the two MRS's in the product formula are equal so it ultimately is given by $L^3(1 - w/L)\sqrt{w/L} = L(L - w)\sqrt{Lw} = 48{,}117$, Friedman's results. Thus, the MRS's implicitly in the product formula map into the equal MRS's implicitly in the Pareto Optimum function. The self-referential principle $x = F(x)$ is again evident.