There are 100 points possible on this exam, 50 points each for Prof. Lozada's questions and Prof. Dugar's questions.

There are three sections on this exam:

- In the first section contains all of the required questions. There are four of them. The first two (from Prof. Lozada) are worth 19 points each; the last two (from Prof. Dugar) are worth 18 points each.
- In the second section there are two questions; you should work one of them. Each is worth 14 points.
- In the third section there are two questions; you should work one of them. Each is worth 12 points.
You have 4 hours and 30 minutes (that is, until $1: 30 \mathrm{PM}$ ) to finish this test. This gives you about 45 minutes per question.

Do not use different colors in your answers because we may grade looking at black-and-white photocopies of your exam.

Answers with illegible or difficult-to-read-handwriting may lower your grade because we may not be able to read and understand your answers, especially considering that we are looking at copies. So it is in your best interest to make your answers LEGIBLE.

Please put the number of the problem you are working on at the top of every page of your answers, so we do not accidentally ignore part of your answer.

In this document some questions begin on one page and end on the next page; therefore, do not assume that a question ends at the bottom of a page, but check to determine whether it continues onto the next page.

If you think there is some ambiguity in a question, spell out exactly how you interpret that question.

A correct answer will not receive full credit without supporting explanations (mathematical, non-mathematical, or both). Also, show all your work to receive full credit.

Good luck.

## Section 1.

## Answer all of the following four questions.

1. [19 points] Consider a cost-minimizing firm which produces an output $y$ using two inputs $x_{1}$ and $x_{2}$ according to the production function $y=f\left(x_{1}, x_{2}\right)$. Make the usual assumptions that the firm takes the prices of $x_{1}$ and $x_{2}$ to be fixed, that both $\partial f / \partial x_{1}$ and $\partial f / \partial x_{2}$ are strictly positive, and that there are diminishing returns to each input.
Call an input "inferior" if when output increases, the firm chooses to use less of this input.
(a) Is it possible for both $x_{1}$ and $x_{2}$ to be inferior (simultaneously)? You should be able to answer this without solving any optimization problem; indeed, undergraduate students who do not know calculus should be able to answer this.
(b) Under what conditions is $x_{1}$ inferior? (This is not a question undergraduates could solve.)
(c) Under what conditions is $x_{2}$ inferior?
(d) Use the answers to (a), (b), and (c) to argue that it is impossible for $f_{12}^{\prime \prime}$ to be very negative.
(e) Could $x_{1}$ or $x_{2}$ ever actually be inferior? When?
2. [19 points] Suppose there are two consumers, Tom and Harry, and each obtains utility from hours of sleep, $s$, and from consuming good $x$. Tom and Harry's utility functions are

$$
u_{t}\left(s_{t}, x_{t}\right)=s_{t} x_{t} \quad \text { and } \quad u_{h}\left(s_{h}, x_{h}\right)=s_{h} x_{h}
$$

respectively. Neither Tom nor Harry has any initial endowment of $x$; their initial endowments of time are

$$
\omega_{t}=1 \quad \text { and } \quad \omega_{h}=2
$$

respectively, and they divide their time into hours of sleep and hours of work " $w$ ". (Be careful not to confuse $w$ and $\omega$ even though the two letters look similar.)
The total amount of good $x$ in the economy is created from the total amount of work $w$ in the economy according to the production function $x=4 w$.
(a) Find the Pareto Optimal allocations of $s_{t}, x_{t}, s_{t}$, and $s_{h}$. Hint: the answer can be written in several forms; one is

$$
\begin{array}{ll}
s_{t}=\frac{1}{2} \alpha & x_{t}=2 \alpha \\
s_{h}=\frac{3}{2}-\frac{1}{2} \alpha & x_{h}=6-2 \alpha
\end{array}
$$

where $\alpha$ is some constant.
(b) Suppose Tom and Harry form a two-person competitive economy. However, suppose there is government interference in this economy: the government subsidizes Tom's purchases of $x$ and taxes Harry's purchases of $x$, using so-called ad valorem ["according to value"] taxes $\tau$. (Note that Tom is poor and Harry is rich because of the differences in their endowments. Also, ignore any constraint concerning the government's budget balance.)
i. Explain why Tom solves the problem

$$
\max _{s_{t}, x_{t}} s_{t} x_{t} \quad \text { s.t. }\left(1-s_{t}\right) p_{w}=p_{x} x_{t}(1-\tau)
$$

and why Harry solves the problem

$$
\max _{s_{h}, x_{h}} s_{h} x_{h} \quad \text { s.t. }\left(2-s_{h}\right) p_{w}=p_{x} x_{h}(1+\tau)
$$

ii. Show that in competitive equilibrium,

$$
s_{t}=\frac{1}{2} \quad \text { and } \quad s_{h}=1 .
$$

iii. Either show that in competitive equilibrium

$$
x_{t}^{*}=\frac{6(1+\tau)}{3-\tau}
$$

or show that in competitive equilibrium

$$
x_{h}^{*}=\frac{12(1-\tau)}{3-\tau}
$$

(you do not have to explain both of these).
iv. From your answers to (a), (b-ii), and (b-iii), explain that the allocation in the two-person competitive economy with distortionary taxes can only be Pareto Optimal if the tax rate $\tau$ is zero.
3. [18 points] Two firms, $i$ and $j$, compete in prices in a homogeneous goods industry. The firms interact in period $t=1,2,3, \ldots$. The firms face a time-sensitive demand for their product given by the function $q_{t}=a_{t}-p_{t} . q_{t}$ is the quantity demanded in period $t . p_{i, t}\left(p_{j, t}\right)$ is the price set by firm $i(j)$ in period $t$. Define $p_{t}=\min \left\{p_{i, t}, p_{j, t}\right\}$ in period $t=1,2,3, \ldots$. In other words, $p_{t}$ is the lower of the two prices offered in a given period, $t$. The demand intercept $a_{t}$ is an i.i.d. random variable that takes the value $\bar{a}$ with probability $\beta$, associated with a 'boom' period, and the value $\underline{a}$ with probability $1-\beta$, associated with a 'bust' period. Note that $\overline{\bar{a}}>\underline{a}$. Both firms observe the realizations of $a_{t}$ before they set prices $p_{i, t}$ and $p_{j, t}$. The cost of production is normalized to zero for both firms. The profit function of firm $i$ in period $t$ can be written as:

$$
\pi_{i \neq j, t}= \begin{cases}p_{i, t}\left(a_{t}-p_{i, t}\right) & p_{i, t}<p_{j, t} \\ 0.5 p_{i, t}\left(a_{t}-p_{i, t}\right) & p_{i, t}=p_{j, t} \\ 0 & p_{i, t}>p_{j, t}\end{cases}
$$

Assume that both firms discount future profits at rate $0<\delta<1$.
(a) Suppose that firms compete only once instead of playing the game repeatedly. Find the equilibrium prices and profits for each of the two firms. Show all your work to receive full credit.
(b) Suppose that firms compete only once instead of playing the game repeatedly. Find the symmetric collusive market prices (i.e., the monopoly prices) and individual firm profits as a function of the parameter $a_{t}$. Show all your work to receive full credit.
(c) Now, suppose that firms repeatedly compete for infinite times (i.e., $t=1,2,3, \ldots$ ). Assume that $\beta=1$. Moreover, both firms use a grim trigger strategy (GTS), as discussed in the class, that allows them to charge forever the monopoly price under $\beta=1$ you found in part (b). What values of $\delta$ can support both firms using the above GTS as a subgame perfect Nash equilibrium (SPNE)? How does your answer change when $\beta=0$ ? Formally write down the GTS that you use in your answer. Show all your work to receive full credit.
(d) Now assume $0<\beta<1$ and firms play an infinitely repeated game, as in part (c). Moreover, both firms use a grim trigger strategy (GTS), as discussed in the class, that allows them to
charge the monopoly price under $0<\beta<1$ forever. What values of $\delta$ can support both firms using the above GTS as a subgame perfect Nash equilibrium (SPNE)? You don't need to write down the GTS. Show all your work to receive full credit.
[Hint: first, calculate the discounted expected profit of charging the monopoly price under $0<\beta<1$ forever, starting in the next period. Your answers from part (b) would be helpful for that calculation.]
(e) Is the lower bound on $\delta$ in part (d) lower or higher than the lower bound on $\delta$ in part (c)? Support your answer with an appropriate mathematical argument.
4. [18 points] Suppose there are two unselfish persons, Deb (D) and Debi ( Di i , in a community. Each person has the Bernoulli utility function $u_{i}$ and the vNM expected utility function $U_{i}$. Each $U_{i}$ takes the following form: $U_{i}=u_{i}+\alpha u_{j} ; i \neq j, i=\mathrm{D}, \mathrm{Di}$, and where $\alpha \in(0,1)$ is the weight that player $i$ 's vNM expected utility function assigns to player $j$ 's Bernoulli utility. Each person initially faces the probability $\theta$ of contracting the Coronavirus. Each person's probability of contracting the virus is independent of the other person's probability of contracting the virus. If they contract the virus, their Bernoulli utility is $-b$, and if they do not, it is $b>0$. Suppose there is a vaccine that offers $100 \%$ protection from the virus. The vaccine also reduces the probability that the other person, if unvaccinated, contracts the virus from $\theta$ to $\gamma$, where $\gamma<\theta$. However, each person incurs a disutility from getting the vaccine, which is $-c<0$.
In your answers, denote each person's decision to vaccinate as ' $v$ ' and not vaccinate as ' $n v$ '.
(a) Assume that Debi decides not to receive the vaccine. On the other hand, Deb is deciding whether to receive the vaccine or not. Find a condition under which Deb chooses to receive the vaccine. It will be helpful for you to answer some of the following questions if your above condition expresses $c$ in terms of all other parameters. Call this $c, c(a)$. Show all your work to receive full credit.
(b) Redo part (a) assuming that Deb is now a selfish person (i.e., Deb's vNM expected utility function does not assign any positive weight to Debi's Bernoulli utility; $\alpha=0$ ). Every other detail about the strategic situation described in part (a) and above
that remains the same. Again, it will be helpful for you to answer some of the following questions if your above condition expresses $c$ in terms of all other parameters. Call this $c, c(b)$. Show all your work to receive full credit.
(c) Is the upper bound of $c(a)$, call it $c(a)_{\max }$, higher or lower than the upper bound of $c(b)$, call it $c(b)_{\max }$ ? Support your answer with an appropriate mathematical argument. A correct answer without the supporting mathematical detail will not receive any credit.
(d) Suppose that the actual magnitude of $c$, call it $c_{a}$, is such that $c(b)_{\max }<c_{a}<c(a)_{\max }$. Now assume that your aim is to get Deb vaccinated and, furthermore, you have the 'power' to choose between an unselfish Deb or a selfish Deb. Which version of Deb would you choose? Explain why.
(e) Now, suppose that Deb and Debi are both unselfish as before. Hence, work with the vNM utility function that assigns $\alpha \in$ $(0,1)$ weight to the other person's Bernoulli utility function. Both persons are now deciding whether to receive the vaccine or not. Find the conditions that yield the pure strategy Nash equilibrium (PSNE) in which both choose to vaccinate, that is, $(v, v)$. Show all your work to receive full credit.

## Section 2.

## Answer one of the following two questions.

1. [14 points] Consider a bargaining game between two players, a proposer $(\mathrm{P})$ and a responder ( R ). Both players are risk-neutral and must divide $\$ 1$ according to the following procedure: Player P can offer a division of $\$ 1$ of the form $(x, 1-x)$ with $x \in[0,1]$. Let $x$ be player P's share. Player R can accept or reject. The proposed division is implemented if player R accepts and the game ends. If player R rejects the offer, both players receive zero, and the game ends. All of this is common knowledge.
Both players care about their monetary payoff and also about fairness. If the allocation is $\left(x_{P}, x_{R}\right)$, their utilities are:

$$
\begin{aligned}
& u_{P}\left(x_{P}, x_{R}\right)=x_{P}-\theta_{P}\left|x_{P}-x_{R}\right| \\
& u_{R}\left(x_{P}, x_{R}\right)=x_{R}-\theta_{R}\left|x_{P}-x_{R}\right| .
\end{aligned}
$$

Assume that $\theta_{P}, \theta_{R} \geq 0$ are parameters that measure how much players care about fairness.
Find all the pure strategy subgame perfect Nash equilibria of the above game. [Hint: your answer will depend on the parameters $\theta_{P}$ and $\theta_{R}$.]
2. [14 points] Consider the following game played by two risk-neutral players, 1 and 2. Each player has an initial endowment of $y>0$ dollars. The players simultaneously and independently decide how many of those $y$ dollars to contribute to a common pot. Let us denote player $i$ 's contribution decision to the common pot by $c_{i}$, where $0 \leq$ $c_{i} \leq y ; i=1,2$. The monetary payoff to player $i$ from the above game is given by:

$$
\pi_{i}=y-c_{i}+r G \quad \text { for } i=1,2 \text { and } i \neq j .
$$

$r$ is the marginal per capita return to a player, arising from one dollar in the common pot, $G=c_{1}+c_{2}$. In other words, $G$ is the total monetary contributions toward the common pot.
(a) Assume that players are selfish and their utility function is given by: $u_{i}\left(\pi_{i}, \pi_{j}\right)=\pi_{i}$. Derive a condition under which each player chooses to contribute zero dollars (i.e., $c_{i}^{*}=0$ ) to the common
pot in a pure strategy Nash equilibrium (PSNE). Verify that both players contributing zero dollars is indeed a PSNE. Show all your work to receive full credit.
(b) Continue to assume that players are selfish, their utility function is given by: $u_{i}\left(\pi_{i}, \pi_{j}\right)=\pi_{i}$, and the condition you found in part (a) holds. Let us define the social objective function: $W=\sum_{i=1}^{2} \pi_{i}$. Suppose we want to maximize the value of $W$. Derive a condition under which the value of $W$ is maximized, and what is the $W$-maximizing value of $c_{i}$ (call it $c_{i}^{w}$ ). Check if both players contributing $c_{i}^{w}$ to the common pot is a PSNE or not. Show all your work to receive full credit.
(c) Now, suppose that players also care about inequality in monetary payoffs between themselves. Each player's new unselfish utility function is now given by:

$$
\begin{aligned}
& u_{i}\left(\pi_{i}, \pi_{j}\right)=\pi_{i}-\alpha_{i} \max \left\{\pi_{j}-\pi_{i}, 0\right\}-\beta_{i} \max \left\{\pi_{i}-\pi_{j}, 0\right\} \\
& \text { for } i=1,2 \text { and } i \neq j .
\end{aligned}
$$

Assume that $0<\beta_{i}<1$ and $\alpha_{i}>\beta_{i}$.
$\alpha_{i}$ is player $i^{\prime} s$ envy parameter since $\pi_{j}>\pi_{i}$ leads to disadvantageous inequality for player $i$. By contrast, $\beta_{i}$ is player $i^{\prime} s$ guilt parameter since $\pi_{i}>\pi_{j}$ leads to advantageous inequality for player $i$.
Use the new unselfish utility function to write down a sufficient condition under which each player chooses to contribute zero dollars to the common pot. That is, $c_{i}^{*}=0$. You do not need to derive the condition. The condition backed up by a clear economic intuition should be sufficient. Show all your work to receive full credit. [Hint: to answer this part, you can begin by assuming that player $i$ is the lowest contributor between the two players. That is, $c_{i}<c_{j}$.]

## Section 3.

## Answer one of the following two questions.

(In Prof. Lozada's opinion, they are of equal difficulty, and their first four parts are identical.)

1. [12 points] Suppose a consumer gets utility from two commodities, apples $a$ and bananas $b$, according to the utility function

$$
u(a, b)=a b^{2} .
$$

Suppose the consumer takes as given the price of apples, $p_{a}$, and the price of bananas, $p_{b}$, and that the consumer has income denoted by $m$.
(a) Find this consumer's demand for apples and demand for bananas. Do not bother checking the second-order conditions.
(b) For the rest of this problem, suppose that the price of apples is $p_{a}=1 / 3$ and the price of bananas is $p_{b}=2 / 3$.
If this consumer's initial income is $m_{0}=3$, determine his initial consumption of apples and of bananas.
(c) If this consumer's income changes to $m^{\prime}=4$, determine his new consumption of apples and of bananas.
(d) [Optional motivation for the rest of the problem: You might think that all "valuation" methods for the $\$ 1$ in extra income the consumer gets when going from $m_{0}=3$ to $m^{\prime}=4$ would assign this change a value of exactly $\$ 1$, but do they?]
Make a sketch of this consumer's indifference curve through his initial consumption point (the one for $m_{0}=3$ ) and this consumer's indifference curve through his new consumption point (the one for $m^{\prime}=4$ ), graphing apples on the horizontal axis and bananas on the vertical axis. You do not draw this graph very precisely. In particular, you do not need to use the mathematical form of this consumer's actual utility function, $u=a b^{2}$, in making your graph; it is fine to draw your graph in a generic way, using the sort of indifference curves discussed in undergraduate textbooks.
(e) On the graph you just drew, indicate this consumer's "Willingness (and Ability) to Pay," measured in terms of apples, to move from his initial consumption point to his new consumption point. This is equal to this consumer's compensating variation, measured in terms of apples.
(f) Numerically calculate this consumer's Willingness (and Ability) to Pay, measured in terms of apples, to move from his initial consumption point to his new consumption point. To do this, you need to use the fact that the consumer's utility function is $u=a b^{2}$, and you need to use the coordinates in the (apples, bananas) plane of the initial and final consumption points.
Since you do not have calculators, you do not need to simplify expressions that are purely numerical. For example, you would not need to simplify $4-27 / 16$.
(g) In the previous part of this problem, you calculated this consumer's Willingness (and Ability) to Pay, measured in terms of apples, to move from his initial consumption point to his new consumption point. What is the dollar value of this WATP number of apples, using the prevailing price of apples? (Again, you do not need to simplify expressions that are purely numerical.)
(h) On the graph you drew above, indicate this consumer's "Willingness (and Ability) to Pay," measured in terms of bananas, to move from his initial consumption point to his new consumption point. This is equal to this consumer's compensating variation, measured in terms of bananas.
(i) Numerically calculate this consumer's Willingness (and Ability) to Pay, measured in terms of bananas, to move from his initial consumption point to his new consumption point. To do this, you need to use the fact that the consumer's utility function is $u=a b^{2}$, and you need to use the coordinates in the (apples, bananas) plane of the initial and final consumption points.
You do not need to simplify expressions that are purely numerical.
(j) In the previous part of this problem, you calculated this consumer's Willingness (and Ability) to Pay, measured in terms of bananas, to move from his initial consumption point to his new consumption point. What is the dollar value of this WATP number of bananas, using the prevailing price of bananas? (Again, you do not need to simplify expressions that are purely numerical.)
2. [12 points] Suppose a consumer gets utility from two commodities, apples $a$ and bananas $b$, according to the utility function

$$
u(a, b)=a b^{2} .
$$

Suppose the consumer takes as given the price of apples, $p_{a}$, and the price of bananas, $p_{b}$, and that the consumer has income denoted by $m$.
(a) Find this consumer's demand for apples and demand for bananas. Do not bother checking the second-order conditions.
(b) For the rest of this problem, suppose that the price of apples is $p_{a}=1 / 3$ and the price of bananas is $p_{b}=2 / 3$.
If this consumer's initial income is $m_{0}=3$, determine his initial consumption of apples and of bananas.
(c) If this consumer's income changes to $m^{\prime}=4$, determine his new consumption of apples and bananas.
(d) [Optional motivation for the rest of the problem: You might think that all "valuation" methods for the $\$ 1$ in extra income the consumer gets when going from $m_{0}=3$ to $m^{\prime}=4$ would assign this change a value of exactly $\$ 1$, but do they?]
Make a sketch of this consumer's indifference curve through his initial consumption point (the one for $m_{0}=3$ ) and this consumer's indifference curve through his new consumption point (the one for $m^{\prime}=4$ ), graphing apples on the horizontal axis and bananas on the vertical axis. You do not draw this graph very precisely. In particular, you do not need to use the mathematical form of this consumer's actual utility function, $u=a b^{2}$, in making your graph; it is fine to draw your graph in a generic way, using the sort of indifference curves discussed in undergraduate textbooks.
(e) On the graph you just drew, indicate this consumer's "Willingness to Accept" compensation, measured in terms of apples, in return for being denied the opportunity to move from his initial consumption point to his new consumption point. This is equal to this consumer's equivalent variation, measured in terms of apples.
(f) Numerically calculate this consumer's Willingness to Accept compensation, measured in terms of apples, in return for being denied the opportunity to move from his initial consumption point
to his new consumption point. To do this, you need to use the fact that the consumer's utility function is $u=a b^{2}$, and you need to use the coordinates in the (apples, bananas) plane of the initial and final consumption points.
Since you do not have calculators, you do not need to simplify expressions that are purely numerical. For example, you would not need to simplify $4-27 / 16$.
(g) In the previous part of this problem, you calculated this consumer's Willingness to Accept compensation, measured in terms of apples, in return for being denied the opportunity to move from his initial consumption point to his new consumption point. What is the dollar value of this WTA number of apples, using the prevailing price of apples? (Again, you do not need to simplify expressions that are purely numerical.)
(h) On the graph you drew above, indicate this consumer's "Willingness to Accept" compensation, measured in terms of bananas, in return for being denied the opportunity to move from his initial consumption point to his new consumption point. This is equal to this consumer's equivalent variation, measured in terms of bananas.
(i) Numerically calculate this consumer's Willingness to Accept compensation, measured in terms of bananas, in return for being denied the opportunity to move from his initial consumption point to his new consumption point. To do this, you need to use the fact that the consumer's utility function is $u=a b^{2}$, and you need to use the coordinates in the (apples, bananas) plane of the initial and final consumption points.
You do not need to simplify expressions that are purely numerical.
(j) In the previous part of this problem, you calculated this consumer's Willingness to Accept compensation, measured in terms of bananas, in return for being denied the opportunity to move from his initial consumption point to his new consumption point. What is the dollar value of this WTA number of bananas, using the prevailing price of bananas? (Again, you do not need to simplify expressions that are purely numerical.)

